1/f noise: a pedagogical review.

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1. Introduction.

The triode was invented by Lee de Forest in 1907, and soon afterwards the first amplifiers were built. By 1921 the "thermionic tube" amplifiers were so developed that C. A. Hartmann [1] made the first courageous experiment to verify Schottky's formula for the shot noise spectral density [2]. Hartmann's attempt failed, and it was finally J. B. Johnson who successfully measured the predicted white noise spectrum [3]. However Johnson also measured an unexpected "flicker noise" at low frequency: his results are shown in figure 1, and shortly thereafter W. Schottky tried to provide a theoretical explanation [4].

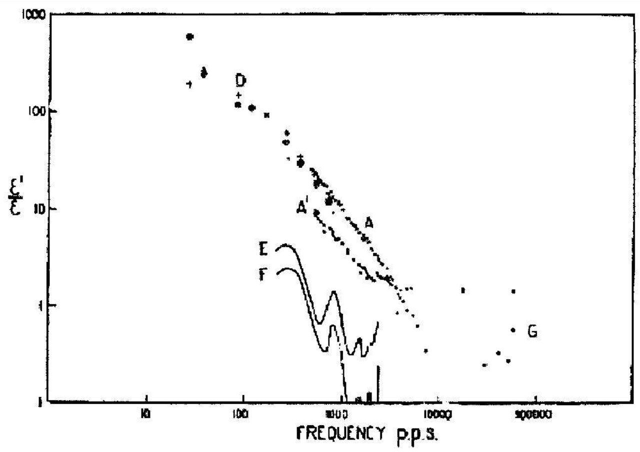

Fig. 6. Frequency variation for tube No. 2, coated filament; same data as in Fig. 4 plotted to a frequency scale; curves E and F give Hartmann's results for 2 m-a. and 20 m-a.; points G were obtained with less steady measuring circuit.

Figure 1: the spectral density observed by J. B. Johnson, as shown in his original paper [3]. The vertical scale represents the observed noise power density divided by the theoretical shot noise power density; the horizontal scale is the frequency in Hz.

Schottky's explanation was based on the physics of electron transport inside the vacuum tube, but in the years that followed Johnson's discovery of flicker noise it was found that this strange noise appeared again and again in many different electrical devices.
The observed spectral density of flicker noise is actually quite variable: it behaves like $1/f^{\alpha}$, where α is in the range $0.5 \div 1.5$, and usually this behavior extends over several frequency decades.
The appearance of power laws in the theory of critical phenomena and above all the work of B. Mandelbrot on fractals in the 1970's [5], seemed to indicate that something deeper was hidden in those ubiquitous spectra. Power laws and *1/f* spectra were found most unexpectedly in many different phenomena, and figure 2 shows two such spectra reproduced in a famous review paper by W. H. Press [6].

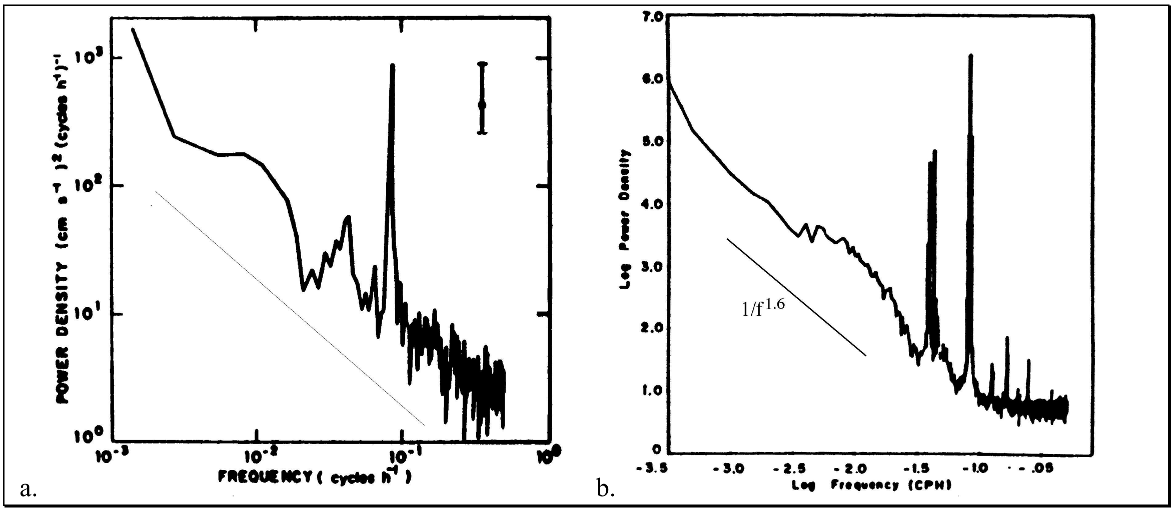

Figure 2: a. power spectrum of the east-west component of ocean current velocity [7]; the straight line shows the slope of a *1/f* spectrum. b. sea level at Bermuda: this is $1/f^{\alpha}$ spectrum with $\alpha \approx 1.6$ [8].

The work of Clarke and Voss on *1/f* noise in resistors also spawned an interesting aside, a study of *1/f* noise in music, which become widely known thanks to an excellent popularization made by M. Gardner in his *Scientific American* column [9]. Clarke and Voss found that both voice and music broadcasts have *1/f* spectra (see figure 3) [10], and they even devised an algorithm to compose "fractal" music [11].

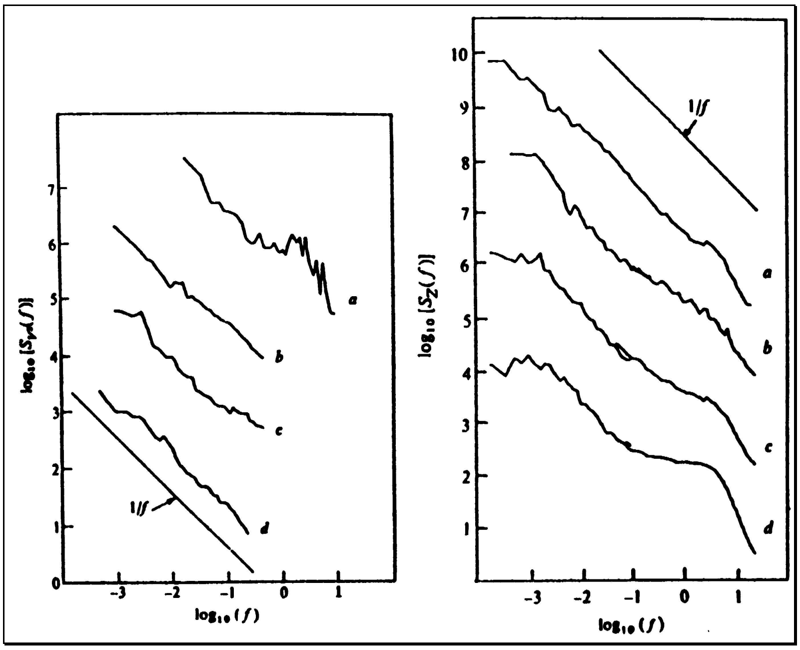

Figure 3: Loudness (left) and pitch (right) fluctuation spectra vs. frequency (Hz) (log-log scale), for a. Scott Joplin piano rags; b. classical radio station; c. rock station; d. news-and-talk station. (from ref. [10]).

By then many physicists were convinced that there had to be a deep reason for the ubiquity of this kind of power-law noises, that there had to be something akin to the universality of exponents in critical phenomena, and therefore many people set out to find an all-encompassing explanation.

2. *$1/f^\alpha$* noise from the superposition of relaxation processes

An early and simple explanation of the appearance of $1/f^\alpha$ noise in vacuum tubes was implicit in some comments of Johnson [3], and was stated mathematically by Schottky [4]: there is a contribution to the vacuum tube current from cathode surface trapping sites, which release the electrons according to a simple exponential relaxation law $N(t) = N_0 e^{-\lambda t}$ for $t \geq 0$ and $N(t) = 0$ for $t < 0$. The Fourier transform of a single exponential relaxation process is

$$F(\omega) = \int_{-\infty}^{+\infty} N(t) e^{-i\omega t} dt = N_0 \int_0^{+\infty} e^{-(\lambda + i\omega)t} dt = \frac{N_0}{\lambda + i\omega} \tag{1}$$

therefore for a train of such pulses $N(t,t_k) = N_0 e^{-\lambda(t-t_k)}$ for $t \geq t_k$ and $N(t,t_k) = 0$ for $t < t_k$, we find

$$F(\omega) = \int_{-\infty}^{+\infty} \sum_k N(t,t_k)\, e^{-i\omega t} dt = N_0 \sum_k e^{i\omega t_k} \int_0^{+\infty} e^{-(\lambda+i\omega)t} dt = \frac{N_0}{\lambda + i\omega} \sum_k e^{i\omega t_k} \tag{2}$$

and the spectrum is

$$S(\omega) = \lim_{T\to\infty} \frac{1}{T} \left\langle |F(\omega)|^2 \right\rangle = \frac{N_0^2}{\lambda^2 + \omega^2} \lim_{T\to\infty} \frac{1}{T} \left\langle \left| \sum_k e^{i\omega t_k} \right|^2 \right\rangle = \frac{N_0^2 n}{\lambda^2 + \omega^2} \tag{3}$$

where n is the average pulse rate and the triangle brackets denote an ensemble average. This spectrum is nearly flat near the origin, and after a transition region it becomes proportional to $1/\omega^2$ at high frequency. This was sufficient for Schottky, who had found such a dependence in Johnson's data, but later it became clear that a single relaxation process was not enough, and that there had to be a superposition of such processes, with a distribution of relaxation rates λ [12]. If the relaxation rate is uniformly distributed between two values λ_1 and λ_2, and the amplitude of each pulse remains constant, we find the spectrum

$$\begin{aligned} S(\omega) &= \frac{1}{\lambda_2 - \lambda_1} \int_{\lambda_1}^{\lambda_2} \frac{N_0^2 n}{\lambda^2 + \omega^2} d\lambda = \frac{N_0^2 n}{\omega(\lambda_2 - \lambda_1)} \left[\arctan\frac{\lambda_2}{\omega} - \arctan\frac{\lambda_1}{\omega} \right] \\ &\approx \begin{cases} N_0^2 n & 0 < \omega \ll \lambda_1 \ll \lambda_2 \\ \dfrac{N_0^2 n \pi}{2\omega(\lambda_2 - \lambda_1)} & \lambda_1 \ll \omega \ll \lambda_2 \\ \dfrac{N_0^2 n}{\omega^2} & \lambda_1 \ll \lambda_2 \ll \omega \end{cases} \end{aligned} \tag{4}$$

(see figures 4 and 5).

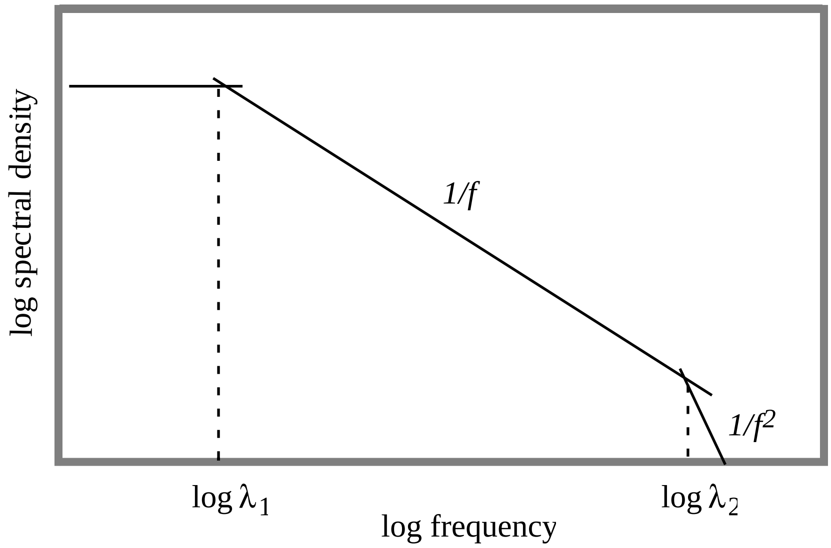

Figure 4: schematic shape of the spectral density (4). There are three characteristic regions: a white noise region at very low frequency, a *1/f* noise intermediate region and a *1/f*^{2} region at high frequency.

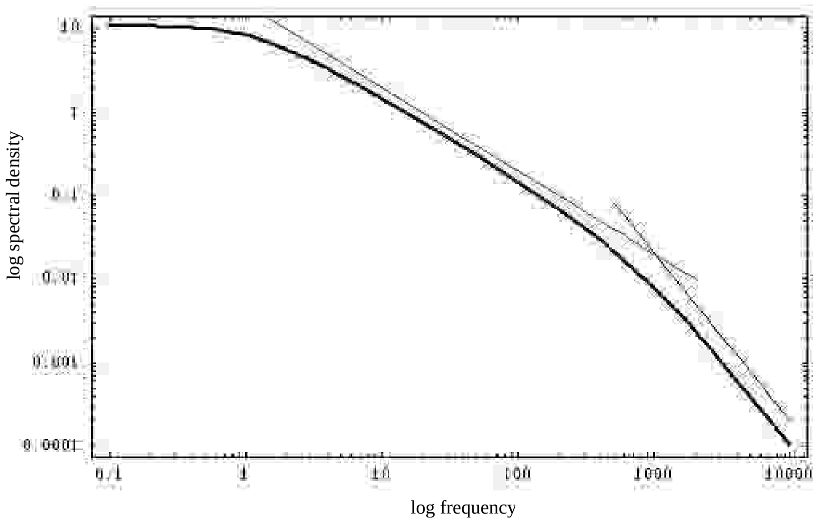

Figure 5: spectral density (arbitrary units) obtained from the superposition of 10000 relaxation processes with decay rates uniformly distributed and equally spaced between 1 and 1000 (arbitrary frequency units). The straight lines represent a *1/f* (red) and a *1/f*^{2} (blue) spectral density.

How stable is this result? Is it still possible to obtain similar spectra changing the assumptions to fit reasonable physical needs?

These questions can at least partly be answered with a direct numerical simulation: figure 6 shows the result for a uniform random distribution of the relaxation rates, and it is clear that the final distribution is insensitive to small deviations from a perfectly uniform distribution of the relaxation rates.

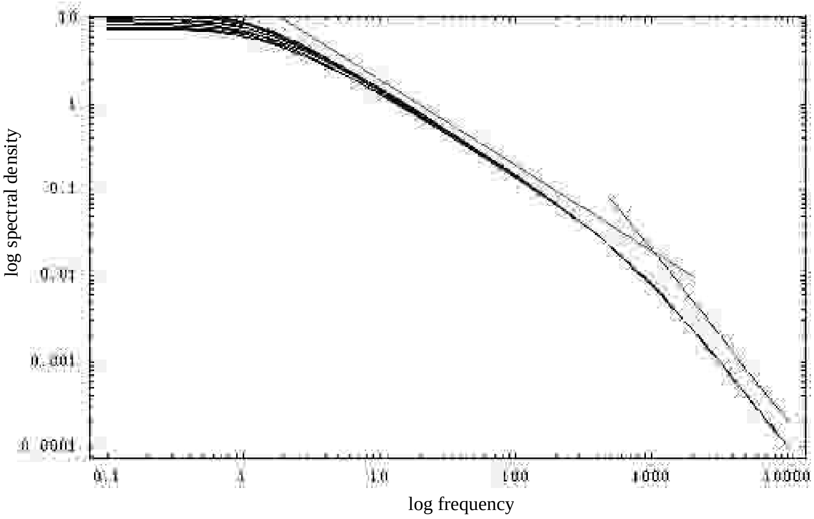

Figure 6: spectral densities (arbitrary units) obtained from the superposition of 10000 relaxation processes with decay rates randomly and uniformly distributed between 1 and 1000 (arbitrary frequency units). 10 spectral densities were generated and superposed on the same plot (black solid lines). The straight lines represent a *1/f* (red) and a *1/f²* (blue) spectral density.

We shall see later that the relaxation rates may be distributed according to different distributions, for instance we may have

$$dP(\lambda) = \frac{A}{\lambda^{\beta}}\, d\lambda \tag{5}$$

in the range $\lambda_1 < \lambda < \lambda_2$: in this case it is still possible to integrate the spectrum exactly[1] and we obtain

$$S(\omega) \propto \int_{\lambda_1}^{\lambda_2} \frac{1}{\lambda^2+\omega^2}\frac{d\lambda}{\lambda^{\beta}} = \begin{cases} \left.\dfrac{1}{\omega^2}\ln\dfrac{\lambda}{\sqrt{\lambda^2+\omega^2}}\right|_{\lambda_1}^{\lambda_2} & \text{if} \quad \beta = 1 \\ \left.\dfrac{\lambda^{1-\beta}}{(1-\beta)\omega^2} F\left(\dfrac{1-\beta}{2}, 1; 1+\dfrac{1-\beta}{2}; -\dfrac{\lambda^2}{\omega^2}\right)\right|_{\lambda_1}^{\lambda_2} & \text{if} \quad \beta \neq 1 \end{cases} \tag{6}$$

where

$$F(a,b;c;z) = {}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty}\frac{(a)_k (b)_k}{(c)_k}\frac{z^k}{k!} = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt \tag{7}$$

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[1] Similar calculations have been done by Butz [13] and are summarized in the review paper by van der Ziel [14]

is the usual hypergeometric function. However we do not have to use the exact expression (6) to find the behavior of the spectral density in the range $\lambda_1 << \lambda << \lambda_2$, since we can approximate the exact integral as follows:

$$\begin{aligned} S(\omega) &\propto \int_{\lambda_1}^{\lambda_2} \frac{1}{\lambda^2+\omega^2}\frac{d\lambda}{\lambda^\beta} = \frac{1}{\omega^{1+\beta}}\int_{\lambda_1}^{\lambda_2} \frac{1}{\left(1+\lambda^2/\omega^2\right)}\frac{d(\lambda/\omega)}{(\lambda/\omega)^\beta} = \frac{1}{\omega^{1+\beta}}\int_{\lambda_1/\omega}^{\lambda_2/\omega}\frac{1}{\left(1+x^2\right)}\frac{dx}{x^\beta} \\ &\approx \frac{1}{\omega^{1+\beta}}\int_0^\infty \frac{1}{\left(1+x^2\right)}\frac{dx}{x^\beta} \propto \frac{1}{\omega^{1+\beta}} \end{aligned} \tag{8}$$

and thus we obtain a whole class of flicker noises with different exponents.

3. Infinitely large fluctuations?

From the previous discussion one may argue that it is important to find experimentally the actual limiting values λ_1 and λ_2, in order to characterize the noise process. Unfortunately this is seldom possible, and in most cases it seems that the *1/f* behavior continues as far as one can see: consider for instance the beautiful data of Pellegrini, Saletti, Terreni and Prudenziati[15] shown in figure 7, the *1/f* behavior extends over more than 6 frequency decades and there seems to be still no noise power flattening at low frequency.

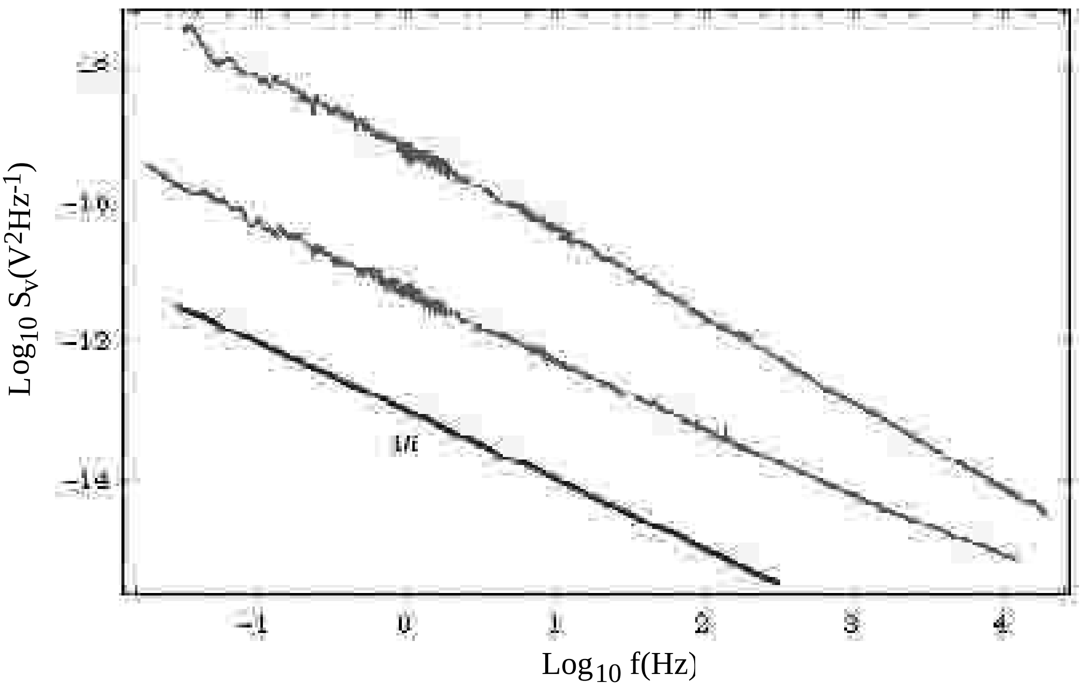

Figure 7. This figure shows the spectral densities of voltage fluctuations measured in two thin-film resistors by Pellegrini, Saletti, Terreni and Prudenziati [15]. They are very close to a perfect *1/f* noise, the behavior extends over more than 6 frequency decades and there seems to be no noise power flattening at low frequency.